

Physics 1A
Spring 2018

Practice Final

Problem 1

Let the muzzle velocity of a projectile be v_0 and the angle of elevation be θ . Calculate the projectile's speed and range. Assuming that the force caused by air resistance is directly proportional to the projectile's velocity, i.e. $\mathbf{F}_r = km\mathbf{v}$, find the position of the object as a function of time.

Solution 1

Using the $\mathbf{F} = m\mathbf{g}$, the force components become

$$0 = m\ddot{x} \quad \text{in the } x\text{-direction} \quad (1)$$

and

$$-mg = m\ddot{y} \quad \text{in the } y\text{-direction} \quad (2)$$

Assume that at $t = 0, x = y = 0$. Then

$$\begin{aligned} \ddot{x} &= 0 \\ \dot{x} &= v_0 \cos \theta \\ x &= v_0 t \cos \theta \end{aligned} \quad (3)$$

and

$$\begin{aligned} \ddot{y} &= -g \\ \dot{y} &= -gt + v_0 \sin \theta \\ y &= \frac{-gt^2}{2} + v_0 t \sin \theta \end{aligned} \quad (4)$$

The speed as functions of time is

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{v_0^2 + g^2 t^2 - 2v_0 g t \sin \theta} \quad (5)$$

We can find the range by determining the value of x when the projectile falls back to ground, that is, when $y = 0$.

$$y = t \left(\frac{-gt}{2} + v_0 \sin \theta \right) = 0 \quad (6)$$

One value of $y = 0$ occurs for $t = 0$ and the other one for $t = T$.

$$\frac{-gT}{2} + v_0 \sin \theta = 0 \quad \longrightarrow \quad T = \frac{2v_0 \sin \theta}{g} \quad (7)$$

The range R is the found from

$$x(t = T) = \text{range} = \frac{2v_0^2}{g} \sin \theta \cos \theta \quad \longrightarrow \quad R = \frac{v_0^2}{g} \sin 2\theta \quad (8)$$

As before, the initial conditions are

$$\begin{aligned} x(t = 0) &= 0 = y(t = 0) \\ \dot{x}(t = 0) &= v_0 \cos \theta \equiv U \\ \dot{y}(t = 0) &= v_0 \sin \theta \equiv V \end{aligned} \quad (9)$$

In the presence of friction, the equations of motion are

$$\begin{aligned} m\ddot{x} &= -km\dot{x} \\ m\ddot{y} &= -km\dot{y} - mg \end{aligned} \quad (10)$$

The equation of motion in the x -direction is solved in class and the solution is

$$x = \frac{U}{k}(1 - e^{-kt}) \quad (11)$$

Similarly, the equation of motion in the y -direction is also solved in class and the solution is

$$y = -\frac{gt}{k} + \frac{kV + g}{k^2}(1 - e^{-kt}) \quad (12)$$

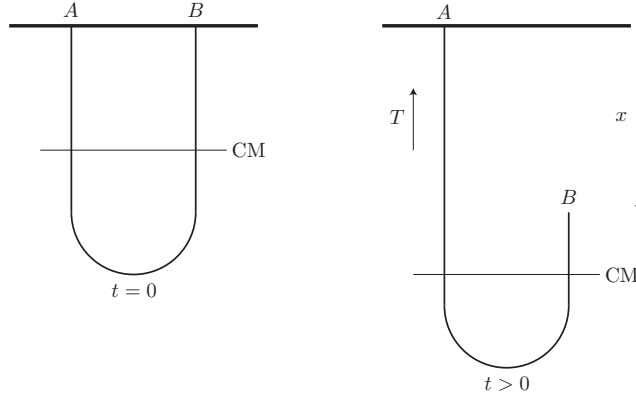
Problem 2

A chain of uniform linear mass density ρ , length b , and mass M ($\rho = M/b$) hangs from the ceiling. At time $t = 0$, the ends A and B are adjacent, but end B is released. Find the tension in the chain at point A after end B has fallen a distance x by assuming free fall.

Solution 2

Let's assume the only forces acting on the system at time t are the tension T acting vertically upward at point A and the gravitational force Mg pulling the chain down. The center of mass momentum reacts to these forces such that

$$\dot{P} = Mg - T \quad (13)$$



The right side of the chain, with mass $\rho(b-x)/2$, is moving at the speed $\dot{x}$, and the left side of the chain is not moving. The total momentum of the system is therefore

$$P = \rho \left(\frac{b-x}{2} \right) \dot{x} \quad (14)$$

and

$$\dot{P} = \frac{\rho}{2} [-\dot{x}^2 + \ddot{x}(b-x)] \quad (15)$$

For free fall, we have $x = gt^2/2$, so that

$$\begin{aligned} \dot{x} &= gt = \sqrt{2gx} \\ \ddot{x} &= g \end{aligned} \quad (16)$$

and

$$\dot{P} = \frac{\rho}{2} (gb - 3gx) = Mg - T \quad (17)$$

and finally

$$T = \frac{Mg}{2} \left(\frac{3x}{b} + 1 \right) \quad (18)$$

Problem 3

A projectile of mass M and velocity $\mathbf{v}$ explodes while in flight into three fragments. One mass ($m_1 = M/2$) travels in the original direction of the projectile, mass m_2 ($= M/6$) travels in the opposite direction, and mass m_3 ($= M/3$) comes to rest. The explosion releases an energy E which is equal to five times the projectile's kinetic energy just before the explosion happens. What are the velocities of the fragments in terms of $\mathbf{v}$?

Solution 3

Let the velocity of the projectile of mass M be $\mathbf{v}$. The three fragments have the following masses and velocities:

$$\begin{aligned} m_1 &= \frac{M}{2}, & \mathbf{v}_1 &= k_1 \mathbf{v} & \text{Forward direction, } & k_1 > 0 \\ m_2 &= \frac{M}{6}, & \mathbf{v}_2 &= -k_2 \mathbf{v} & \text{Opposite direction, } & k_2 > 0 \\ m_3 &= \frac{M}{3}, & \mathbf{v}_3 &= 0 & \text{At rest} \end{aligned} \quad (19)$$

The conservation of linear momentum and energy give

$$\begin{aligned} Mv &= \frac{M}{2}k_1v - \frac{M}{6}k_2v \\ E + \frac{1}{2}Mv^2 &= \frac{1}{2}\frac{M}{2}(k_1v)^2 + \frac{1}{2}\frac{M}{6}(k_2v)^2 \end{aligned} \quad (20)$$

From the conservation of momentum equation, $k_2 = 3k_1 - 6$, which we can insert into the conservation of energy equation and obtain:

$$5 \left(\frac{1}{2}Mv^2 \right) + \frac{1}{2}Mv^2 = \frac{Mv^2}{4}k_1^2 + \frac{Mv^2}{12}(3k_1 - 6)^2 \quad (21)$$

which reduces to $k_1^2 - 3k_1 = 0$, giving the results $k_1 = 0$ and $k_1 = 3$. For $k_1 = 0$, the value of $k_2 = -6$, which is inconsistent with $k_2 > 0$. For $k_1 = 3$, the value of $k_2 = 3$. The velocities become

$$\begin{aligned} \mathbf{v}_1 &= 3\mathbf{v} \\ \mathbf{v}_2 &= -3\mathbf{v} \\ \mathbf{v}_3 &= 0 \end{aligned} \quad (22)$$

Problem 4

A uniform, solid sphere of mass M and radius R and a uniform, solid cylinder of the same mass and radius are released simultaneously at rest from the top of an inclined plane of length ℓ that makes an angle θ with the horizontal. Which reaches the bottom first if they both roll without slipping?

Solution 4

Using conservation of kinetic plus potential energy from the top to the bottom of the inclined plane (there is no net non-conservative work done by the force of static friction), we find that

$$Mg(\ell \sin \theta + R \cos \theta) = \frac{1}{2}MV^2 + \frac{1}{2}I\omega^2 + MgR \cos \theta \quad (23)$$

where we have assigned the bottom of the inclined plane to be $y = 0$, V is the speed of the cylinder at the bottom of the incline, and ω is its angular speed at the bottom of the incline.

Notice also that we have included an $MgR\cos\theta$ term on either side of the conservation equation due to the fact that the center of mass of the cylinder is actually a distance $\ell\cos\theta$ above the plane at any given point, but this term cancels on both sides, so it could have been omitted from the beginning. We also note that since the cylinder is rolling without slipping, we have the constraint

$$V = R\omega \quad (24)$$

Plugging this into the conservation of energy equation and solving for V gives

$$v = R\sqrt{\frac{2Mgd\sin\theta}{MR^2 + I}} \quad (25)$$

to determine the velocity of the cylinder at the bottom, we simply plug in the moment of inertia of the cylinder which is

$$I = \frac{1}{2}MR^2 \quad (26)$$

and we get

$$v_{\text{cyl}} = \sqrt{\frac{4}{3}g\ell\sin\theta} \quad (27)$$

Doing the same computation with the moment of inertia of the sphere gives

$$v_{\text{sph}} = \sqrt{\frac{10}{7}g\ell\sin\theta} \quad (28)$$

The acceleration of the center of mass of each object is constant along the inclined plane, so the one that has the higher final speed is the one that gets to the bottom first. We determine which is larger by taking their ratio;

$$\frac{v_{\text{sph}}}{v_{\text{cyl}}} = \sqrt{\frac{10}{7} \frac{3}{4}} = \sqrt{\frac{15}{14}} > 1. \quad (29)$$

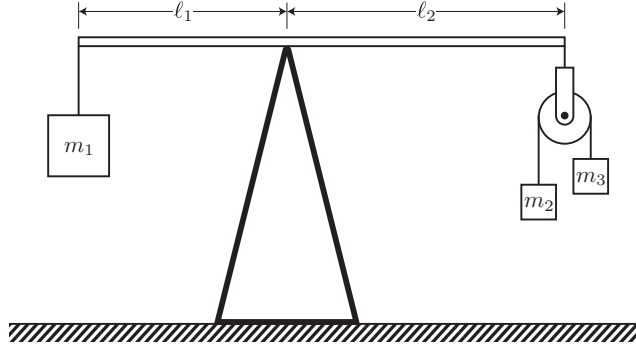
Hence the sphere gets to the bottom faster.

Problem 5

A pivoted beam has mass m_1 suspended from one end and an Atwood's machine suspended from the other with masses m_2 and m_3 suspended on either side. The frictionless pulley has negligible mass and size.

(a) Find the relation between m_1 , m_2 , m_3 , l_1 , and l_2 which will ensure that the beam has no tendency to rotate just after the masses are released.

(b) What would you predict the relation would be in the case that all three masses are equal? Does your answer from part (a) agree with that prediction?



Solution 5

We take the location of the vertex of the fulcrum to be the origin and let the positive x-direction point to the right and the positive y-direction point vertically upward. The tension in the string holding mass m_1 is m_1g , so that tension exerts a torque on the left hand side equal to

$$\vec{\tau}_L = m_1gl_1\hat{k} \quad (30)$$

On the right hand side, denoting the tension connecting the masses T , noting that the accelerations of the two masses are equal and opposite, and applying Newton's Second Law in the y-direction to each mass gives

$$T - m_2g = m_2a, \quad T - m_3g = -m_3a \quad (31)$$

On the other hand, applying Newton's second Law to the pulley, and using the fact that it's massless, we find that the tension T' in the rope holding the pulley satisfies

$$T' - 2T = 0 \quad (32)$$

These equations can be solved for the tension T' to give

$$T' = 4g \frac{m_2m_3}{m_2 + m_3} \quad (33)$$

The torque due to the string on the right hand side is therefore

$$\vec{\tau}_R = -T'l_2\hat{k} = -4g \frac{m_2m_3}{m_2 + m_3} l_2\hat{k} \quad (34)$$

Since the angular momentum of this system doesn't change while it's in equilibrium, the net external torque is zero;

$$\vec{\tau}_L + \vec{\tau}_R = 0 \quad (35)$$

which gives the desired relationship:

$$\boxed{m_2 l_1 = 4 \frac{m_2 m_3}{m_2 + m_3} l_2} \quad (36)$$

When all three masses are equal (say to m), the tension in the left string will be mg while the tension on the right string will be $2mg$ (there will be no accelerations of the right-hand masses in this case). For the torques to balance, the tension that is twice as small needs to be acting twice as far away on the rod to compensate, so I would expect the relation to reduce to $l_1 = 2l_2$. In this special case, the relationship we just derived gives

$$ml_1 = 4 \frac{m^2}{2m} l_2 \quad (37)$$

and this implies $l_1 = 2l_2$ as expected! Can you think of any more limiting cases to check?

Problem 6

A man of mass M stands on a railroad car which is rounding an unbanked turn of radius R at speed v . His center of mass is a height L above the car, and his feet are a distance d apart. How much weight is on each of his feet?

Solution 6

We first note that the rate of change of L' , the angular momentum of the man relative to his center of mass, is zero¹, and this implies that the net external torque about this center of mass is zero:

$$\vec{\tau}'_{\text{ext}} = 0. \quad (38)$$

The external forces on the man are the normal forces on his feet, his weight, and the friction on his feet keeping his body moving in a circle. Let N_1 be the magnitude of the normal force on his inner foot, let N_2 be the magnitude of the normal force on his outer foot, and let f denote the magnitude of the friction force, Newton's

¹This is not actually as obvious as it might seem at first. It's clear that he's not rotating about his center of mass in the plane containing his arms and legs, but if you were to watch him from above from a frame of reference fixed relative to the track, you would see that he is spinning around his center of mass along an axis passing vertically through his center of mass. Nonetheless, this spinning occurs at a constant rate, so the rate of change of the spinning, and hence the associated angular momentum, is zero.

Second Law in the x- and y-directions reveals the following relationships between these various forces.

$$\begin{aligned} N_1 + N_2 &= Mg \\ f &= M \frac{v^2}{R} \end{aligned} \tag{39}$$

Here we have assumed that d is much smaller than R , so we can treat the radius at which the force of friction acts as R for both of his feet, despite the fact that his inner foot is at radius R and his outer foot is at radius $R + d$. Setting the sum of the external torques about this center of mass to zero gives

$$-fL - N_1 \frac{d}{2} + N_2 \frac{d}{2} = 0 \tag{40}$$

Note that gravity can be treated as acting at his center of mass, so it exerts no torque about his center of mass. We thus have three equations in three unknowns, N_1 , N_2 , and f . We are after N_1 and N_2 since these are the contact forces between each of his feet and the platform on which he stands. Some algebra gives

$$\boxed{N_1 = \frac{Mg}{2} - \frac{Mv^2}{R} \frac{L}{d}, \quad N_2 = \frac{Mg}{2} + \frac{mv^2}{R} \frac{L}{d}} \tag{41}$$